

# $SL(2, R)$ symmetry and quasi-normal modes in the BTZ black hole

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**ABSTRACT:** With the help of two new intrinsic tensor fields associated with the  $SL(2, R)$  quadratic Casimir of Killing fields, we uncover the  $SL(2, R)$  symmetry satisfied by the solutions to the equations of motion for various fields in the BTZ black hole in a uniform way by performing tensor and spinor analysis without resorting to any specific coordinate system. Then with the standard algebraic method developed recently, we determine the quasi-normal modes for various fields in the BTZ black hole. As a result, the quasi-normal modes are given by the infinite tower of descendants of the chiral highest weight mode, which is in good agreement with the previous analytic result obtained by exactly solving equations of motion instead.

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## 1. Introduction

As a concrete implementation of holographic principle, AdS/CFT correspondence is definitely one of the most significant developments in fundamental physics. With this holographic dictionary, the black hole in the bulk is dual to the conformal field theory at a finite temperature. In particular, as first suggested and numerically tested in [1], the quasi-normal modes associated with the linear perturbation of the black hole correspond to the operators perturbing the dual field theory at the thermal equilibrium, and the quasi-normal frequencies arise as the poles of the retarded Green function of the operators. Due to the simplicity of  $AdS_3/CFT_2$ , where exact computations can be performed on both sides, the precise quantitative agreement has been confirmed for scalar, fermion, and vector perturbations of the BTZ black hole by solving the equations of motion analytically[2]. Later, in the spirit of [3] and [4], it is demonstrated that the quasi-normal modes can be constructed as an infinite tower of descendants of left and right chiral highest weight mode associated with the  $SL(2, R)$  Lie algebra of Killing fields for scalar and tensor perturbations of the BTZ black hole[5]. More recently, such an algebraic construction has been generalized to other three dimensional black holes with the vector perturbation included[6, 7]. However, the intermediate calculation

involved there is a little bit complicated, which makes the  $SL(2, R)$  symmetry lost in our eyes although it is uncovered in the final step somehow. The technical reason for that comes partially from the fact that the ordinary derivative is used there rather than the covariant derivative. It is highly expected that the computation will be simplified once one adopts the covariant derivative instead since it is intrinsic to the spacetime geometry.

The purpose of this paper is two fold. One is to develop the simplified computational strategy by making transparent the  $SL(2, R)$  symmetry during the course of our analysis. To achieve this, besides appealing to the covariant derivative as alluded to above, we also introduce two intrinsic tensor fields associated with the  $SL(2, R)$  symmetry. It turns out that these two guys have very nice properties and always conspire to organize and manipulate the calculation as simple as possible. The other is to incorporate the fermion field into our analysis in a uniform way.

The rest of paper is structured as follows. In the next section, after a brief review of the  $SL(2, R)$  Lie algebra of Killing fields in the BTZ black hole and its quadratic Casimir operator, we introduce the two intrinsic tensor fields associated with the quadratic Casimir and uncover their intriguing features. With this preparation, we provide an explicit derivation of how the solutions to equations of motion fall into the representation of the  $SL(2, R)$  Lie algebra for various fields in the BTZ black hole in Section 3. Then in the subsequent section, we construct the corresponding quasi-normal modes by the standard algebraic approach. As expected, the result is in good agreement with the previous computation. The conclusion and discussion are put into the last section.

Notation and conventions follow [8] unless specified otherwise.

## 2. $SL(2, R)$ quadratic Casimir in the BTZ black hole

### 2.1 Two sets of $SL(2, R)$ Lie algebra associated with the Killing fields

Without loss of generality, let us start with the non-rotating BTZ black hole with unit mass, i.e.,

$$ds^2 = -\sinh^2(\rho)d\tau^2 + \cosh^2(\rho)d\varphi^2 + d\rho^2. \quad (2.1)$$

In what follows, we would like to work in the light cone coordinates, i.e.,  $u = \tau + \varphi, v = \tau - \varphi$ , in which the metric takes the form

$$g_{ab} = \frac{1}{4}\{(du)_a(du)_b - \cosh(2\rho)[(du)_a(dv)_b + (du)_b(dv)_a] + (dv)_a(dv)_b\} + (d\rho)_a(d\rho)_b. \quad (2.2)$$

Whence the inverse metric can be obtained as

$$g^{ab} = -\frac{4}{\sinh^2(2\rho)}\{(\frac{\partial}{\partial u})^a(\frac{\partial}{\partial u})^b + \cosh(2\rho)[(\frac{\partial}{\partial u})^a(\frac{\partial}{\partial v})^b + (\frac{\partial}{\partial u})^b(\frac{\partial}{\partial v})^a]\}$$

$$+(\frac{\partial}{\partial v})^a(\frac{\partial}{\partial v})^a\}+(\frac{\partial}{\partial \rho})^a(\frac{\partial}{\partial \rho})^b, \quad (2.3)$$

and the associated volume element reads

$$\epsilon = \frac{\sinh(2\rho)}{4} du \wedge dv \wedge d\rho. \quad (2.4)$$

Note that the above BTZ black hole is locally  $AdS_3$ . In particular, the Riemann and Ricci tensors are given by

$$R_{abcd} = g_{ad}g_{bc} - g_{ac}g_{bd}, R_{ab} = -2g_{ab}. \quad (2.5)$$

Thus such a black hole also admits six Killing fields. A Killing field  $\xi$ , by definition, is a vector field which can generate one-parameter group of isometries, or equivalently, a vector field satisfying the Killing equation  $\nabla_a \xi_b = \nabla_{[a} \xi_{b]}$  with  $\nabla_a$  the covariant derivative operator. With this, one can easily show that the Lie derivative with respect to any Killing field kills all intrinsic tensor fields associated with the metric such as the volume element and commutes with the covariant derivative operator. Here we denote these six Killing fields by  $L_k$  and  $\bar{L}_k$  with  $k = 0, \pm 1$ . In particular,  $L_k$  is given by

$$\begin{aligned} L_0^a &= -(\frac{\partial}{\partial u})^a, \\ L_{-1}^a &= e^{-u}[-\frac{\cosh(2\rho)}{\sinh(2\rho)}(\frac{\partial}{\partial u})^a - \frac{1}{\sinh(2\rho)}(\frac{\partial}{\partial v})^a - \frac{1}{2}(\frac{\partial}{\partial \rho})^a], \\ L_{+1}^a &= e^u[-\frac{\cosh(2\rho)}{\sinh(2\rho)}(\frac{\partial}{\partial u})^a - \frac{1}{\sinh(2\rho)}(\frac{\partial}{\partial v})^a + \frac{1}{2}(\frac{\partial}{\partial \rho})^a]. \end{aligned} \quad (2.6)$$

Similarly,  $\bar{L}_k$  is defined as (2.6) simply by switching  $u$  and  $v$  therein. Locally their Lie commutators satisfy two sets of the  $SL(2, R)$  Lie algebra, i.e.,

$$[L_0, L_{\pm 1}] = \mp L_{\pm 1}, [L_{+1}, L_{-1}] = 2L_0, [\bar{L}_0, \bar{L}_{\pm 1}] = \mp \bar{L}_{\pm 1}, [\bar{L}_{+1}, \bar{L}_{-1}] = 2\bar{L}_0. \quad (2.7)$$

Note that the Lie derivative conforms to  $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]}$  and  $\mathcal{L}_{\alpha X} = \alpha \mathcal{L}_X$  for the arbitrary vector fields  $X$  and  $Y$  with the arbitrary constant  $\alpha$ <sup>1</sup>. Thus the above Lie algebra can be naturally represented by the Lie derivative. In particular, the quadratic Casimir operators can be realized by the Lie derivative as

$$\mathcal{L}^2 = \mathcal{L}_{L_0} \mathcal{L}_{L_0} - \frac{1}{2}(\mathcal{L}_{L_{+1}} \mathcal{L}_{L_{-1}} + \mathcal{L}_{L_{-1}} \mathcal{L}_{L_{+1}}), \bar{\mathcal{L}}^2 = \mathcal{L}_{\bar{L}_0} \mathcal{L}_{\bar{L}_0} - \frac{1}{2}(\mathcal{L}_{\bar{L}_{+1}} \mathcal{L}_{\bar{L}_{-1}} + \mathcal{L}_{\bar{L}_{-1}} \mathcal{L}_{\bar{L}_{+1}}), \quad (2.8)$$

which commute with  $\mathcal{L}_{L_k}$  and  $\mathcal{L}_{\bar{L}_k}$ .

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<sup>1</sup>When the Lie derivative acts on the spinor, the situation will become a little bit subtle. In particular,  $[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_{[X, Y]}$  holds only for the case in which either  $X$  or  $Y$  is the conformal Killing field[9]. Fortunately this subtlety does not bother us as we focus only on the Lie derivative along Killing fields here.

## 2.2 Two types of tensor fields associated with the quadratic Casimir

Now we would like to construct the two types of tensor fields associated with the quadratic  $SL(2, R)$  Casimir as follows

$$H^{ab} = L_0^a L_0^b - \frac{1}{2}(L_{+1}^a L_{-1}^b + L_{-1}^a L_{+1}^b), \bar{H}^{ab} = \bar{L}_0^a \bar{L}_0^b - \frac{1}{2}(\bar{L}_{+1}^a \bar{L}_{-1}^b + \bar{L}_{-1}^a \bar{L}_{+1}^b), \quad (2.9)$$

and

$$\begin{aligned} Z_{abc} &= L_{0a} \nabla_b L_{0c} - \frac{1}{2}(L_{+1a} \nabla_b L_{-1c} + L_{-1a} \nabla_b L_{+1c}), \\ \bar{Z}_{abc} &= \bar{L}_{0a} \nabla_b \bar{L}_{0c} - \frac{1}{2}(\bar{L}_{+1a} \nabla_b \bar{L}_{-1c} + \bar{L}_{-1a} \nabla_b \bar{L}_{+1c}). \end{aligned} \quad (2.10)$$

Apparently  $H$  and  $\bar{H}$  are symmetric tensor fields and a straightforward calculation further gives

$$H^{ab} = \bar{H}^{ab} = \frac{1}{4}g^{ab}. \quad (2.11)$$

Concerning  $Z$  and  $\bar{Z}$  fields, we firstly notice that they are antisymmetric with respect to the last two indices due to the Killing equation. On the other hand, it follows from  $\nabla_b H_{ac} = \nabla_b \bar{H}_{ac} = 0$  that they are also antisymmetric with respect to the first and third indices. Therefore  $Z$  and  $\bar{Z}$  are actually totally antisymmetric tensor fields and should be proportional to the three dimensional volume element. Furthermore, we have

$$\begin{aligned} \nabla^a Z_{abc} &= L_{0a} \nabla^a \nabla_b L_{0c} - \frac{1}{2}(L_{+1a} \nabla^a \nabla_b L_{-1c} + L_{-1a} \nabla^a \nabla_b L_{+1c}) \\ &= R_{cb}{}^{ad} [L_{0a} L_{0d} - \frac{1}{2}(L_{+1a} L_{-1d} + L_{-1a} L_{+1d})] = 0. \end{aligned} \quad (2.12)$$

Here we have used the Killing equation in the first step and the identity

$$\nabla_a \nabla_b \xi_c = R_{cba}{}^d \xi_d \quad (2.13)$$

for any Killing field  $\xi$  in the second step. In addition, in the last step we have employed the fact that Riemann tensor satisfies  $R_{abcd} = R_{ab[cd]}$ . Likewise, we also have  $\nabla^a \bar{Z}_{abc} = 0$ . Whence the proportional coefficients in front of the volume element should be constant. In particular, the explicit calculation yields

$$Z_{abc} = \frac{1}{4}\epsilon_{abc}, \bar{Z}_{abc} = -\frac{1}{4}\epsilon_{abc}, \quad (2.14)$$

which is consistent with the prevalent claim made in the previous literature, namely, the two sets of  $SL(2, R)$  Lie algebra have the opposite chirality.

### 3. $SL(2, R)$ symmetry for various fields in the BTZ black hole

#### 3.1 Tensor fields

As a warm-up, let us start with the scalar field  $\phi$ , whose equation of motion is given by

$$(\nabla_a \nabla^a - m^2)\phi = 0. \quad (3.1)$$

By definition, the Lie derivative acting on the scalar field gives

$$\mathcal{L}_X \mathcal{L}_Y \phi = X^a \nabla_a (Y^b \nabla_b \phi) = (X^a \nabla_a Y^b) \nabla_b \phi + X^a Y^b \nabla_a \nabla_b \phi. \quad (3.2)$$

Whence it is easy to show

$$\mathcal{L}^2 \phi = Z^a {}^b \nabla_b \phi + H^{ab} \nabla_a \nabla_b \phi = \frac{1}{4} g^{ab} \nabla_a \nabla_b \phi = \frac{m^2}{4} \phi. \quad (3.3)$$

Similarly, we have

$$\bar{\mathcal{L}}^2 \phi = \frac{m^2}{4} \phi. \quad (3.4)$$

Now let us move onto the massive vector field  $A$  with equation of motion given by

$$\epsilon_a {}^{bc} \nabla_b A_c = -m A_a. \quad (3.5)$$

Whereby we can obtain

$$\nabla_a A^a = -\frac{1}{m} \epsilon^{abc} \nabla_a \nabla_b A_c = -\frac{1}{m} \epsilon^{abc} R_{abc} {}^d A_d = -\frac{1}{m} \epsilon^{abc} R_{[abc]} {}^d A_d = 0, \quad (3.6)$$

where the cyclic identity  $R_{[abc]d} = 0$  has been used in the last step. On the other hand, we have

$$\begin{aligned} m^2 A^d &= -m \epsilon^{dea} \nabla_e A_a = \epsilon^{dea} \nabla_e (\epsilon_a {}^{bc} \nabla_b A_c) = \epsilon^{ade} \epsilon_a {}^{bc} \nabla_e \nabla_b A_c \\ &= (g^{dc} g^{eb} - g^{db} g^{ec}) \nabla_e \nabla_b A_c = \nabla_a \nabla^a A^d - \nabla_a \nabla^d A^a \\ &= \nabla_a \nabla^a A^d + \nabla^d \nabla_a A^a - \nabla_a \nabla^d A^a = \nabla_a \nabla^a A^d + R^{dabc} A_c g_{ab} \\ &= \nabla_a \nabla^a A^d - R^{dc} A_c. \end{aligned} \quad (3.7)$$

Acting on this vector field by the Lie derivative, we have

$$\begin{aligned} \mathcal{L}_X \mathcal{L}_Y A_a &= X^b \nabla_b (\mathcal{L}_Y A_a) + \mathcal{L}_Y A_b \nabla_a X^b \\ &= X^b \nabla_b (Y^c \nabla_c A_a + A_c \nabla_a Y^c) + (Y^c \nabla_c A_b + A_c \nabla_b Y^c) \nabla_a X^b \\ &= (X^b \nabla_b Y^c) \nabla_c A_a + X^b Y^c \nabla_b \nabla_c A_a + (X^b \nabla_a Y^c) \nabla_b A_c + A_c X^b \nabla_b \nabla_a Y^c \\ &\quad + (Y^c \nabla_a X^b) \nabla_c A_b + A_c \nabla_b (Y^c \nabla_a X^b) - A_c Y^c \nabla_b \nabla_a X^b. \end{aligned} \quad (3.8)$$

Whence we can obtain

$$\begin{aligned}
\mathcal{L}^2 A_a &= Z^b{}_c \nabla_b \nabla_c A_a + H^{bc} \nabla_b \nabla_c A_a + 2Z^c{}_a \nabla_c A_b + A_c \nabla_b Z^c{}_a + A^c R_{cabd} H^{bd} - A_c R_{ad} H^{dc} \\
&= \frac{1}{4} g^{bc} \nabla_b \nabla_c A_a - \frac{1}{2} \epsilon_a{}^{cb} \nabla_c A_b - \frac{1}{4} R_{ac} A^c \\
&= \frac{1}{4} (m^2 + 2m) A_a,
\end{aligned} \tag{3.9}$$

where we have used the identity (2.13) in the first step. Due to the opposite chirality, we have

$$\bar{\mathcal{L}}^2 A_a = \frac{1}{4} (m^2 - 2m) A_a. \tag{3.10}$$

We conclude this subsection by involving ourselves into the massive graviton field  $h$ . The equation of motion is given by

$$\epsilon_a{}^{bc} \nabla_b h_{cd} = -m h_{ad}. \tag{3.11}$$

From this equation, it is easy to find  $g^{ab} h_{ab} = 0$ . In addition, we can obtain

$$\begin{aligned}
\nabla^a h_{ad} &= -\frac{1}{m} \epsilon^{abc} \nabla_a \nabla_b h_{cd} = -\frac{1}{m} \epsilon^{abc} (R_{abc}{}^e h_{ed} + R_{abd}{}^e h_{ce}) \\
&= -\frac{1}{m} \epsilon^{abc} (\delta_a^e g_{bd} - g_{ad} \delta_b^e) h_{ce} = -\frac{1}{m} \epsilon^{abc} (g_{bd} h_{ca} - g_{ad} h_{cb}) = 0.
\end{aligned} \tag{3.12}$$

Furthermore, we have

$$\begin{aligned}
m^2 h^e{}_d &= -m \epsilon^{efa} \nabla_f h_{ad} = \epsilon^{efa} \nabla_f (\epsilon_a{}^{bc} \nabla_b h_{cd}) = \epsilon^{aef} \epsilon_a{}^{bc} \nabla_f \nabla_b h_{cd} \\
&= (g^{ec} g^{fb} - g^{eb} g^{fc}) \nabla_f \nabla_b h_{cd} = \nabla_b \nabla^b h^e{}_d - \nabla^c \nabla^e h_{cd} \\
&= \nabla_b \nabla^b h^e{}_d + \nabla^e \nabla^c h_{cd} - \nabla^c \nabla^e h_{cd} \\
&= \nabla_b \nabla^b h^e{}_d - R^{ef} h_{fd} - R^{eaf}{}_d h_{af}.
\end{aligned} \tag{3.13}$$

Now acting on this massive graviton field, the Lie derivative yields

$$\begin{aligned}
\mathcal{L}_X \mathcal{L}_Y h_{ab} &= X^c \nabla_c \mathcal{L}_Y h_{ab} + 2 \mathcal{L}_Y h_{cb} \nabla_a X^c \\
&= X^c \nabla_c (Y^d \nabla_d h_{ab} + 2 h_{db} \nabla_a Y^d) + 2 (Y^d \nabla_d h_{cb} + h_{db} \nabla_c Y^d + h_{cd} \nabla_b Y^d) \nabla_a X^c \\
&= (X^c \nabla_c Y^d) \nabla_d h_{ab} + X^c Y^d \nabla_c \nabla_d h_{ab} + 2 (X^c \nabla_a Y^d) \nabla_c h_{db} + 2 h_{db} X^c \nabla_c \nabla_a Y^d \\
&\quad + 2 (Y^d \nabla_a X^c) \nabla_d h_{cb} + 2 h_{db} \nabla_c (Y^d \nabla_a X^c) - 2 h_{db} Y^d \nabla_c \nabla_a X^c \\
&\quad + 2 h_{cd} \nabla_b (Y^d \nabla_a X^c) - 2 h_{cd} Y^d \nabla_b \nabla_a X^c,
\end{aligned} \tag{3.14}$$

where the symmetrization between the indices  $a$  and  $b$  is implicitly assumed for convenience. Whence we can obtain

$$\mathcal{L}^2 h_{ab} = Z^c{}_d \nabla_d h_{ab} + H^{cd} \nabla_c \nabla_d h_{ab} + 4 Z^c{}_a \nabla_c h_{db} + 2 h_{db} \nabla_c Z^d{}_a + 2 h_{cd} \nabla_b Z^d{}_a$$

$$\begin{aligned}
& +2h_{db}R^d_{ace}H^{ce} - 2h_{db}R_{ae}H^{ed} - 2h_{cd}R^c_{abe}H^{de} \\
& = \frac{1}{4}g^{cd}\nabla_c\nabla_d h_{ab} - \epsilon_a^{cd}\nabla_c h_{db} - \frac{1}{2}R_a^e h_{eb} - \frac{1}{2}R_{aceb}h^{ce} \\
& = \frac{1}{4}(m^2 h_{ab} + 4m h_{ab} - R_a^e h_{eb} - R_{aceb}h^{ce}) \\
& = \frac{1}{4}[m^2 h_{ab} + 4m h_{ab} + 2\delta_a^e h_{eb} - (g_{ab}g_{ce} - g_{ae}g_{cb})h^{ce}] \\
& = \frac{1}{4}(m^2 + 4m + 3)h_{ab}.
\end{aligned} \tag{3.15}$$

By the same token, we have

$$\bar{\mathcal{L}}^2 h_{ab} = \frac{1}{4}(m^2 - 4m + 3)h_{ab}. \tag{3.16}$$

### 3.2 Spinor field

Let us start with Dirac equation

$$(\gamma^a \nabla_a + m)\psi = 0. \tag{3.17}$$

Here  $\gamma^a = e_I^a \gamma^I$  and  $\nabla_a = \partial_a + \frac{1}{4}\omega_{IJ a} \gamma^{IJ}$ , where  $e_I^a$  form a set of orthogonal normal vector bases, and Gamma matrices satisfy  $\{\gamma^I, \gamma^J\} = 2\eta^{IJ}$  with the spin connection  $\omega_{IJ a} = e_{Ib} \nabla_a e_J^b$  and  $\gamma^{IJ} = \frac{1}{2}[\gamma^I, \gamma^J]$ . Acting on both sides of Dirac equation with  $\gamma^b \nabla_b - m$ , we have

$$\begin{aligned}
0 & = (\gamma^b \nabla_b - m)(\gamma^a \nabla_a + m)\psi = (\gamma^a \gamma^b \nabla_a \nabla_b - m^2)\psi \\
& = (g^{ab} \nabla_a \nabla_b + \gamma^{ab} \nabla_a \nabla_b - m^2)\psi = (\nabla_a \nabla^a - m^2)\psi + \gamma^{ab} \nabla_{[a} \nabla_{b]} \psi,
\end{aligned} \tag{3.18}$$

where  $\gamma^{ab} = e_I^a e_J^b \gamma^{IJ}$ . To proceed, we notice

$$\begin{aligned}
\nabla_{[a} \nabla_{b]} \psi & = \partial_{[a} \nabla_{b]} \psi - \Gamma^c_{[ab]} \nabla_c \psi + \frac{1}{4}\omega_{IJ[a} \gamma^{IJ} \nabla_{b]} \psi \\
& = \partial_{[a} \partial_{b]} \psi + \frac{1}{4}[\partial_{[a} (\omega_{MNb]} \gamma^{MN} \psi) + \omega_{IJ[a} \gamma^{IJ} \partial_{b]} \psi + \frac{1}{4}\omega_{IJ[a} \gamma^{IJ} \omega_{MNb]} \gamma^{MN} \psi] \\
& = \frac{1}{4}[(\partial_{[a} \omega_{MNb]}) \gamma^{MN} \psi + \omega_{IJ[b} \gamma^{IJ} \partial_{a]} \psi + \omega_{IJ[a} \gamma^{IJ} \partial_{b]} \psi + \frac{1}{4}\omega_{IJ[a} \gamma^{IJ} \omega_{MNb]} \gamma^{MN} \psi] \\
& = \frac{1}{4}[(\partial_{[a} \omega_{MNb]}) \gamma^{MN} \psi + \frac{1}{4}\omega_{IJ[a} \omega_{MNb]} \gamma^{IJ} \gamma^{MN} \psi] \\
& = \frac{1}{4}\{(\partial_{[a} \omega_{MNb]}) \gamma^{MN} \psi + \frac{1}{8}\omega_{IJ a} \omega_{MNb} [\gamma^{IJ}, \gamma^{MN}]\} \\
& = \frac{1}{4}\{(\partial_{[a} \omega_{MNb]}) \gamma^{MN} \psi + \frac{1}{4}\omega_{IJ a} \omega_{MNb} (\eta^{JM} \gamma^{IN} - \eta^{JN} \gamma^{IM} - \eta^{IM} \gamma^{JN} + \eta^{IN} \gamma^{JM}) \\
& = \frac{1}{4}[(\partial_{[a} \omega_{MNb]}) \gamma^{MN} \psi + \omega_{MIa} \omega^I_{Nb} \gamma^{MN} \psi] = \frac{1}{4}(\partial_{[a} \omega_{MNb]} + \omega_{MI[a} \omega^I_{Nb]}) \gamma^{MN} \psi \\
& = \frac{1}{8}R_{abMN} \gamma^{MN} \psi,
\end{aligned} \tag{3.19}$$



where  $\Gamma$  is the Christoffel symbol and the second Cartan equation has been used in the last step with  $R_{abMN} = R_{abcd}e_M^c e_N^d$ . With this observation, we end up with

$$(\nabla_a \nabla^a - m^2 + \frac{1}{8} R_{abcd} \gamma^{ab} \gamma^{cd}) \psi = 0. \quad (3.20)$$

Note that the Lie derivative acting on spinor fields is given by

$$\mathcal{L}_X \psi = X^a \nabla_a \psi - \frac{1}{4} \gamma^{ab} \psi \nabla_b X_a. \quad (3.21)$$

Thus we have

$$\begin{aligned} \mathcal{L}_X \mathcal{L}_Y \psi &= X^a \nabla_a \mathcal{L}_Y \psi - \frac{1}{4} \gamma^{ab} \mathcal{L}_Y \psi \nabla_b X_a \\ &= X^a \nabla_a (Y^c \nabla_c \psi - \frac{1}{4} \gamma^{cd} \psi \nabla_d Y_c) - \frac{1}{4} \gamma^{ab} (Y^c \nabla_c \psi - \frac{1}{4} \gamma^{cd} \psi \nabla_d Y_c) \nabla_b X_a \\ &= (X^a \nabla_a Y^c) \nabla_c \psi + X^a Y^c \nabla_a \nabla_c \psi - \frac{1}{4} \gamma^{cd} \psi X^a \nabla_a \nabla_d Y_c - \frac{1}{4} (X^a \nabla_a Y_c) \gamma^{cd} \nabla_d \psi \\ &\quad - \frac{1}{4} (Y^c \nabla_b X_a) \gamma^{ab} \nabla_c \psi + \frac{1}{16} \gamma^{ab} \gamma^{cd} \psi \nabla_d (Y_c \nabla_b X_a) - \frac{1}{16} \gamma^{ab} \gamma^{cd} \psi Y_c \nabla_d \nabla_b X_a. \end{aligned} \quad (3.22)$$

Whence it is not hard to show

$$\begin{aligned} \mathcal{L}^2 \psi &= Z^a {}^c \nabla_c \psi + H^{ac} \nabla_a \nabla_c \psi - \frac{1}{4} \gamma^{cd} \psi R_{cdae} H^{ae} - \frac{1}{2} Z^a {}_{dc} \gamma^{cd} \nabla_a \psi \\ &\quad + \frac{1}{16} \gamma^{ab} \gamma^{cd} \psi \nabla_d Z_{cba} - \frac{1}{16} \gamma^{ab} \gamma^{cd} \psi R_{abde} H^e{}_c \\ &= \frac{1}{4} (\nabla_a \nabla^a \psi - \frac{1}{2} \epsilon^a{}_{bc} \gamma^{cb} \nabla_a \psi + \frac{1}{16} R_{abcd} \gamma^{ab} \gamma^{cd} \psi) \\ &= \frac{1}{4} [(m^2 - \frac{1}{16} R_{abcd} \gamma^{ab} \gamma^{cd}) \psi - \frac{1}{2} \epsilon^a{}_{bc} \gamma^{cb} \nabla_a \psi] \\ &= \frac{1}{4} [(m^2 + \frac{1}{8} \gamma_{cd} \gamma^{cd}) \psi + \frac{1}{2} \epsilon^a{}_{bc} \gamma^{bc} \nabla_a \psi] \\ &= \frac{1}{4} (m^2 - \frac{3}{4} + m) \psi, \end{aligned} \quad (3.23)$$

where the identity special to three dimension  $\gamma^{ab} = \epsilon^{abc} \gamma_c$  has been used in the last step. Similarly, we have

$$\bar{\mathcal{L}}^2 \psi = \frac{1}{4} (m^2 - \frac{3}{4} - m) \psi. \quad (3.24)$$

#### 4. Quasi-normal modes in the BTZ black hole

As a recapitulation, we find that the solutions to the equations of motion for various fields fall into the various representations of  $SL(2, R)$  Lie algebra labeled by the value

of the Casimir, i.e.,

$$\mathcal{L}^2\Phi = \lambda_+\Phi, \bar{\mathcal{L}}^2\Phi = \lambda_-\Phi, \quad (4.1)$$

where  $\lambda_\pm = \frac{m^2}{4}$  for the scalar field,  $\lambda_\pm = \frac{m^2 \pm 2m}{4}$  for the vector field,  $\lambda_\pm = \frac{m^2 \pm 4m + 3}{4}$  for the tensor field, and  $\lambda_\pm = \frac{m^2 \pm m - \frac{3}{4}}{4}$  for our spinor field. With this observation, the quasi-normal modes can be constructed by the standard algebraic approach. Speaking specifically, we start from the highest weight mode which obeys the condition as follows

$$\mathcal{L}^2\Phi_+^{(0)} = \lambda_+\Phi_+^{(0)}, \mathcal{L}_{L_{+1}}\Phi_+^{(0)} = 0, \mathcal{L}_{L_0}\Phi_+^{(0)} = w_+\Phi_+^{(0)}, \quad (4.2)$$

or

$$\bar{\mathcal{L}}^2\Phi_-^{(0)} = \lambda_-\Phi_-^{(0)}, \mathcal{L}_{\bar{L}_{+1}}\Phi_-^{(0)} = 0, \mathcal{L}_{\bar{L}_0}\Phi_-^{(0)} = w_-\Phi_-^{(0)}, \quad (4.3)$$

where

$$\Phi_\pm^{(0)} = e^{-i\omega_\pm^0\tau + ip\phi}\Psi_\pm^{(0)}(\rho) \quad (4.4)$$

with  $\mathcal{L}_{L_0}\Psi_\pm(\rho) = \mathcal{L}_{\bar{L}_0}\Psi_\pm(\rho) = 0^2$ . This implies that  $\omega_\pm^0$  and  $p$  correspond to the frequency and angular momentum of this highest weight mode respectively. By the definition of quasi-normal mode, we require the imaginary part of  $\omega_\pm^0$  to be negative and  $p$  to be real.

Now the resultant quasi-normal modes can be constructed as the infinite tower of descendant modes, i.e.,

$$(\mathcal{L}_{\bar{L}_{-1}}\mathcal{L}_{L_{-1}})^n\Phi_\pm^{(0)} \quad (4.5)$$

with  $n = 0, 1, 2, \dots$ . Employing the commutation relation (2.8), one can show that the conformal weight is given by

$$w_+ = \frac{1 \pm \sqrt{1 + 4\lambda_+}}{2}, w_- = \frac{1 \pm \sqrt{1 + 4\lambda_-}}{2}, \quad (4.6)$$

and the corresponding quasi-normal frequencies can be worked out as

$$\omega_\pm^n = \pm p - 2i(w_\pm + n). \quad (4.7)$$

As expected, the result is in good agreement with the previous calculation[2, 5, 6].

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<sup>2</sup>This sort of expansion can always be achieved. In particular,  $\Psi_\pm^{(0)}(\rho)$  can be regarded as the coordinate component of the tensor fields in the coordinate system  $\{u, v, \rho\}$ . For our spinor field, it denotes the component associated with the choice of the orthogonal normal bases as  $e_0^a = \frac{1}{\sinh(\rho)}(\frac{\partial}{\partial\tau})^a, e_1^a = \frac{1}{\cosh(\rho)}(\frac{\partial}{\partial\varphi})^a$ , and  $e_2^a = (\frac{\partial}{\partial\rho})^a$ .

## 5. Conclusion

Instead of solving the equations of motion analytically, we have constructed the quasinormal modes for various fields in the BTZ black hole and determined its frequencies in a uniform way by invoking the algebraic approach. The result is in good agreement with the previous calculation as it should be. To achieve this, the primary task is to show the solutions to equation of motion fall into the representation of the  $SL(2, R)$  Lie algebra, which is fulfilled by the explicit tensor and spinor analysis without resorting to any specific coordinate system. To make such an analysis as simple as possible, we have introduced two tensor fields intrinsic to the  $SL(2, R)$  Casimir of Killing fields and unveiled their intriguing features by relating them to the metric and volume element respectively. As shown, these two tensor fields conspire to play an important role in organizing and simplifying the relevant tensor and spinor analysis.

We conclude with some generalizations of our work in various directions. Firstly, although the analysis for more general fields are expected to go straightforward, it is interesting to investigate how the quasinormal frequencies are quantitatively related to the mass parameter appearing in the equation of motion. On the other hand, besides the BTZ black hole considered here, there are other somewhat complicated three dimensional black holes such as warped black holes and self-dual warped black holes[10, 11]. It is intriguing to show how these cases can be fitted into our framework such that the relevant spacetime symmetry can be made transparent and the whole calculation can be simplified similarly[12]. In addition, associated with the near horizon geometry, the  $SL(2, R)$  symmetry plays an important role in the context of Kerr/CFT correspondence[13]. Thus it is rewarding to see whether our strategy is also applicable to this higher dimensional spacetime by doing something like  $3 + 1$  decomposition[14]. Finally, it is definitely worthwhile to explore whether our strategy can be extended to the more challenging case of hidden conformal symmetry[15].

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## Appendices

### A Calculation of $Z$ and $\bar{Z}$ fields

By the fact that  $\nabla_a \xi_b = \frac{1}{2}(d\xi)_{ab}$  for any Killing field  $\xi$ , we have

$$\begin{aligned} Z_{abc} = Z_{[abc]} &= \frac{1}{2}[L_{0[a}dL_{0bc]} - \frac{1}{2}(L_{+1[a}dL_{-1]bc} + L_{-1[a}dL_{+1]bc})] \\ &= \frac{1}{6}[L_0 \wedge dL_{0abc} - \frac{1}{2}(L_{+1} \wedge dL_{-1abc} + L_{-1} \wedge dL_{+1abc})], \end{aligned} \quad (5.1)$$

where

$$\begin{aligned} L_{0a} &= g_{ab}L_0^b = -\frac{1}{4}(du)_a + \frac{\cosh(2\rho)}{4}(dv)_a, \\ L_{-1a} &= g_{ab}L_{-1}^b = e^{-u}[\frac{\sinh(2\rho)}{4}(dv)_a - \frac{1}{2}(d\rho)_a], \\ L_{+1a} &= g_{ab}L_{+1}^b = e^u[\frac{\sinh(2\rho)}{4}(dv)_a + \frac{1}{2}(d\rho)_a], \end{aligned} \quad (5.2)$$

and

$$\begin{aligned} (dL_0)_{ab} &= \frac{\sinh(2\rho)}{2}(d\rho)_a \wedge (dv)_b, \\ (dL_{-1})_{ab} &= e^{-u}[-\frac{\sinh(2\rho)}{4}(du)_a \wedge (dv)_b + \frac{\cosh(2\rho)}{2}(d\rho)_a \wedge (dv)_b + \frac{1}{2}(du)_a \wedge (d\rho)_b], \\ (dL_{+1})_{ab} &= e^u[\frac{\sinh(2\rho)}{4}(du)_a \wedge (dv)_b + \frac{\cosh(2\rho)}{2}(d\rho)_a \wedge (dv)_b + \frac{1}{2}(du)_a \wedge (d\rho)_b]. \end{aligned} \quad (5.3)$$

With this, we can finally obtain

$$Z = \frac{\sinh(2\rho)}{16}du \wedge dv \wedge d\rho. \quad (5.4)$$

By the symmetry between  $u$  and  $v$ , we have

$$\bar{Z} = \frac{\sinh(2\rho)}{16}dv \wedge du \wedge d\rho = -\frac{\sinh(2\rho)}{16}du \wedge dv \wedge d\rho. \quad (5.5)$$

### B A little bit of Clifford algebra

Firstly by the identity

$$[A, BC] = \{A, B\}C - B\{A, C\}, \quad (5.6)$$

we have

$$[\gamma^I, \gamma^M \gamma^N] = 2(\eta^{IM} \gamma^N - \eta^{IN} \gamma^M), \quad (5.7)$$

which further gives

$$[\gamma^I, \gamma^{MN}] = 2(\eta^{IM}\gamma^N - \eta^{IN}\gamma^M). \quad (5.8)$$

Next by the Jacobi identity, we have

$$\begin{aligned} [\gamma^{IJ}, \gamma^{MN}] &= \frac{1}{2}[[\gamma^I, \gamma^J], \gamma^{MN}] = \frac{1}{2}([\gamma^I, [\gamma^J, \gamma^{MN}]] - [\gamma^J, [\gamma^I, \gamma^{MN}]] \\ &= 2(\eta^{JM}\gamma^{IN} - \eta^{JN}\gamma^{IM} - \eta^{IM}\gamma^{JN} + \eta^{IN}\gamma^{JM}). \end{aligned} \quad (5.9)$$

### C A little bit of spinor analysis

Associated with the choice of the orthogonal normal bases as  $e_0^a = \frac{1}{\sinh(\rho)}(\frac{\partial}{\partial\tau})^a, e_1^a = \frac{1}{\cosh(\rho)}(\frac{\partial}{\partial\varphi})^a$ , and  $e_2^a = (\frac{\partial}{\partial\rho})^a$ , the non-vanishing spin connection is given by

$$\omega_{02a} = -\omega_{20a} = -\frac{\cosh(\rho)}{2}[(du)_a + (dv)_a], \omega_{12a} = -\omega_{21a} = \frac{\sinh(\rho)}{2}[(du)_a - (dv)_a]. \quad (5.10)$$

Thus we have

$$\begin{aligned} \mathcal{L}_{L_0}\Psi(\rho) &= -(\frac{\partial}{\partial u})^a[\partial_a + \frac{1}{2}(\omega_{02a}\gamma^{02} + \omega_{12a}\gamma^{12})]\Psi(\rho) + \frac{\sinh(2\rho)}{8}\gamma^{ab}\Psi(\rho)(d\rho)_a(dv)_b \\ &= \frac{1}{4}[\cosh(\rho)\gamma^{02} - \sinh(\rho)\gamma^{12}]\Psi(\rho) + \frac{\sinh(2\rho)}{8}\gamma^{IJ}\Psi(\rho)e_I^a e_J^b (d\rho)_a(dv)_b \\ &= \frac{1}{4}[\cosh(\rho)\gamma^{02} - \sinh(\rho)\gamma^{12}]\Psi(\rho) + \frac{\sinh(2\rho)}{8}[\frac{1}{\sinh(\rho)}\gamma^{20} - \frac{1}{\cosh(\rho)}\gamma^{21}]\Psi(\rho) \\ &= 0. \end{aligned} \quad (5.11)$$

Similarly, we can obtain  $\mathcal{L}_{\bar{L}_0}\Psi(\rho) = 0$ .

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